Images and Kernels in Linear Algebra

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In mathematics, there are many different fields of study, including calculus, geometry, algebra and others. Mathematics has been thought of as a universal language, in which the numbers represent letters, codes, directions, and numerous other variables. One particular branch of mathematics that we will examine is linear algebra, which deals with linear equations, vector spaces, matrices, and linear transformations. Much of linear algebra is conceptual and proof based. Originated by the philosopher and mathematician, René Descartes, who began looking at mathematics philosophically in which everything was considered true only after it was proven. Descartes' analytical mathematics can now be used in many natural sciences, along with mathematics. Images and kernels, for example, are two components of linear algebra used in engineering to solve for the optimum design of linear circuits in communication systems and for actual ground speed using vector computation of true air and wind speed.

First, let us understand basic definitions of linear algebra. Linear equations are composed of two components to a function, the domain and co-domain. The image of a linear transformation is defined as the set of values corresponding to the domain. The image is defined by definition 3.1.1 in Bretscher on page 128, which states :

Consider a function f from X to Y.

Then the image (f) = {values the function f takes in Y}

$= \{f(x) : x \text{ in } X\}$

$= \{y \text{ in } Y: y = f(x), \text{ for some } x \text{ in } X\}$.

The image (f) is a subset of the codomain Y of f.

In simpler terms, the image is the set of all outputs from a linear transformation. Linear transformations are interpreted as a set of vectors multiplied by a coefficient matrix, which is denoted by $T(\mathbf{x}) = A\mathbf{x}$. When this equation is equal to the zero vector, the vectors for $\mathbf{x}$ are called the kernel. The kernel of T is the solution set of the linear system $A\mathbf{x} = \mathbf{0}$. (Bretscher p. 134)

The definition of a span must also be understood. A span can be thought of as the length between points or extremities. In general, the span of vectors is the set of all linear combinations of the vectors from an initial point to an infinite number or certain value. The span of a set of vectors is acquired by multiplying a given set of vectors by arbitrary scalars. The definition of a vector span can be better explained by considering the vectors $v_1, v_2, \ldots, v_n$ in $\mathbf{R}^m$, in which the set of all multiples of a given vector is also the linear combinations of the vectors $v_1, v_2, \ldots, v_n$. An example of spanning is when vectors of a matrix consist of x, y, and z components. The span $(v_1, v_2, \ldots, v_n) = \{c_1v_1 + c_2v_2 + \ldots + c_nv_n\}$ where c_i are arbitrary scalars. (Bretscher p. 131)

Let us consider a problem from quiz 4, type II, number 5, as an example of an image and a kernel of matrices:

Give an example of matrices A, B such that

a) The image (A) is spanned by the vector $\begin{bmatrix}5\\7\\-3\end{bmatrix}$, and

b) The kernel (B) is spanned by the vector $\begin{bmatrix}5\\7\\-3\end{bmatrix}$.

For part (a) of the problem, we are to find matrix A, such that the image of A is spanned by the vector $[5\ \ 7\ \ -3]^T$.[1]

The Fact 3.1.4 in Bretscher, on page 133 illustrates,

The image of a linear transformation T (from $\mathbf{R}^n$ to $\mathbf{R}^m$) has the following properties:

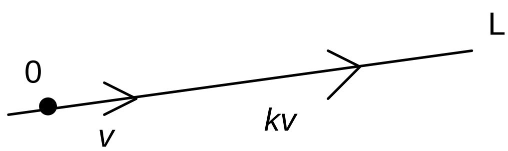
a. The zero vector $\mathbf{0}$ in $\mathbf{R}^m$ is contained in im(T),

b. The image is closed under addition: If $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are both in im(T), then so is $\boldsymbol{v_1} + \boldsymbol{v_2}$,

c. The image is closed under scalar multiplication: If a vector $\boldsymbol{v}$ is in im(T) and k is an arbitrary scalar, then $k\boldsymbol{v}$ is in the image as well.

To interpret these properties, we will examine the image geometrically. If $\boldsymbol{v}$ is in the image, so are all vectors on the line L spanned by $\boldsymbol{v}$, which are scalar multiples of the vector $\boldsymbol{v}$.

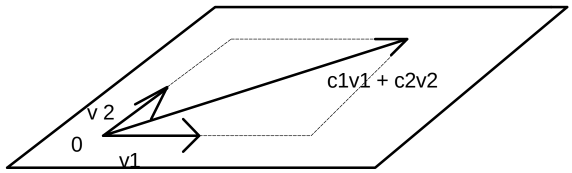
Figure 1.



[1] $[5\ \ 7\ \ -3]^T$ is the transpose of the vector for the problem.

Also, if there are two vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ that are in the same image, then the plane that the two vectors create contain all vectors that span $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.

Figure 2.



The image of a matrix is equal to the span of columns in that matrix. However, if there are vectors that are a linear combination of other vectors within the span, we can exclude the redundant ones. The vectors within the span are then linearly independent because they do not rely on others within the span.

Now that we have established the properties of an image and a span, we can say the vector $[5\ \ 7\ \ -3]^T$ is linearly independent within the image of matrix A.

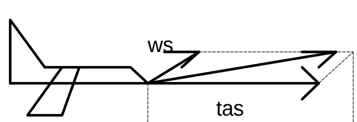
So, matrix A is made up of linear combinations or linearly dependent vectors of the vector $[5\ \ 7\ \ -3]^T$ and has an infinite number of solutions.

One solution is matrix (A) $= \begin{array}{c} \boxed{\begin{matrix} 5 & 10 & -15 \\ 7 & 14 & -21 \\ -3 & -6 & 9 \end{matrix}} \\ \begin{matrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} \end{matrix} \end{array}$

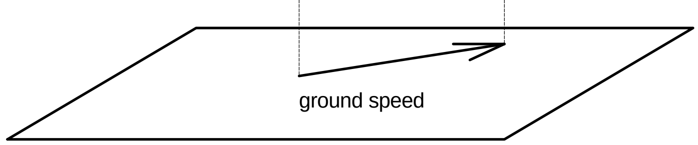
where $\boldsymbol{v_1}$ is multiplied by arbitrary scalars 2 and –3 to create the vectors $\boldsymbol{v_2}$, and $\boldsymbol{v_3}$ respectively. If matrix A has the columns $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, $\boldsymbol{v_3}$, we can say that the vectors $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ are linear combinations of $\boldsymbol{v_1}$, which makes the vectors $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ linearly dependent on $\boldsymbol{v_1}$. Therefore, the image of matrix A is spanned by the vector $[5\ \ 7\ \ -3]^T$. This is only one simple solution to the problem, however; there can be many different solutions.

Images are not only useful in linear algebra, but can also be applied in real life situations. Engineers use images with orthogonal vectors to determine the speed of airplanes. Flight management computers used in airplanes are designed by engineers to use the true air speed vector (tas) measured on the airplane and add them to the wind speed vector (ws) sensed in flight, to produce a resultant vector.

Figure 3.



This resultant vector is the ground speed the image projects on the ground from the nose



of the airplane. Thus, the image of the transformation involving true air speed and wind speed vectors in $\mathbf{R}^3$ is projected onto the $\mathbf{R}^2$ plane. Also, from this simulation, the kernel can be determined which would be all scalar multiples of the vectors within the perpendicular component, from the ground to the nose of the airplane.

For part (b), we are to find matrix B with the kernel, spanned by the vector $[5\ \ 7\ \ -3]^T$. We are interested in the zeros of the transformation. The set of solutions consists of the span of vectors defining the kernel. Let us say that we have a function T(**x**) that is transformed into A**x**, where A is a matrix multiplied by the **x** vector. When we set this transformation A**x** equal to the zero vector, A**x** = 0. The kernel is the set of solutions, which is also spanned by a set of vectors. In our problem the kernel of a matrix B is equal to the span of the vector $[5\ \ 7\ \ -3]^T$. We are to find a matrix B that satisfies the span.

So if Ker(B) = span $([5\ \ 7\ \ -3]^T)$ then the set of solutions is the vector multiplied by a scalar *s*, which is given by :

$$\text{Ker (B)} = \left\{ s\,[5\ \ 7\ \ -3]^T : s \in \mathbf{R}^n \right\} \text{ where } s \text{ is a scalar.}$$

We know that the image of A is equal to the vector $[5\ \ 7\ \ -3]^T$ multiplied by a scalar *s*. This creates a transformation A**x**. When we make this transformation equal to a vector **y** also within the image of A, we solve for a new matrix in terms of the vector **y.** So consider a vector **y** within the image of B, if and only if there exists **x** where B**x** = **y**. Consider the matrix A from the

$$\text{Im(A)=} \boxed{\begin{matrix} 5 & 10 & -15 \\ 7 & 14 & -21 \\ -3 & -6 & 9 \end{matrix}} \quad \text{where } A\mathbf{x} = \mathbf{y}.$$

Through Gauss-Jordan elimination, we can transform the matrix into Reduced Row Echelon Form (RREF). Gauss-Jordan elimination is a method of solving linear systems, which was named after the German mathematician Carl Friedrich Gauss and the German engineer Wilhelm Jordan.

Gauss-Jordan elimination is also known as elementary row operations of linear equations, which allows three possible maneuvers of equations in a system. The three steps are to (*i*) interchange any two equations in the system; (*ii*) divide both sides of any equations by the scalar k, where k can be any integer; or (*iii*) replace the jth equation with the sum of the jth equation plus c times the old ith equation, where c is any scalar.

We obtain the matrix B in Reduced Row Echelon Form as:

$$B = \boxed{\begin{matrix} 1 & 2 & -3 & : & {}^{1}/_{5}\, y_1 \\ 0 & 0 & 0 & : & -{}^{7}/_{5}\, y_1 - y_2 \\ 0 & 0 & 0 & : & {}^{3}/_{5}\, y_1 - y_3 \end{matrix}} \quad .$$

Our result for matrix B can only be consistent, meaning it has exactly one solution or an infinite number of solutions, if and only if, $-{}^{7}/_{5}\, \mathbf{y_1} + \mathbf{y_2} = 0$ and ${}^{3}/_{5}\, \mathbf{y_1} + \mathbf{y_3} = 0$. So we can now set up these two equations in a linear system:

$$\begin{aligned} -{}^{7}/_{5}\, \mathbf{y_1} + \mathbf{y_2} \qquad &= 0 \\ {}^{3}/_{5}\, \mathbf{y_1} \qquad + \mathbf{y_3} &= 0. \end{aligned}$$

Since y is a vector, the system put into matrix form is:

$$\boxed{\begin{matrix} -{}^{7}/_{5} & 1 & 0 \\ {}^{3}/_{5} & 0 & 1 \end{matrix}} \quad \mathbf{y} = \mathbf{x},$$

where the kernel of B is spanned by the vector $\boxed{\begin{matrix} 5 \\ 7 \\ -3 \end{matrix}}$,

making the matrix B = $\boxed{\begin{matrix} -{}^{7}/_{5} & 1 & 0 \\ {}^{3}/_{5} & 0 & 1 \end{matrix}}$.

The kernel of matrix B equals the span of the vector $[5 \;\; 7 \;\; -3]^T$, which also has three properties, given by Fact 3.1.6 in Bretscher on page 138.

Properties of the kernel:

a. The zero vector $\mathbf{0}$ in $\mathbf{R}^n$ is contained in ker (T).
b. The kernel is closed under sums.
c. The kernel is closed under scalar multiples.

These three properties are very similar to the properties of the image.

In conclusion, we have examined the process of finding an image and a kernel of a linear transformation, which can be used, for many practical situations. And since we have a linear transformation that has the same properties of a subspace, the image and kernel of the linear transformation are subspaces of $\mathbf{R}^n$.